MATH 1010E Lecture Notes Week 13 (Martin Li)

Last time ... Partial Fractions.

e.g. $\int \frac{p(x)}{q(x)}\,dx$ where $p(x), q(x)$ are polynomials.

Procedures to solve:

Step 1: Factorize $q(x)$ as much as possible.

$$q(x) = (x-a)^k \underbrace{(x^2+Ax+B)}_{\text{e.g. } x^2+1}.$$

Step 2: Write down the possible terms.

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} + \frac{Cx+D}{x^2+Ax+B}$$

assume $\deg p(x) < \deg q(x)$ (otherwise do long division)

Step 3: Expand the numerator on the R.H.S. and then compare coefficients with $p(x)$.

t-substitution:

Consider an integral of the form

$$\int \underbrace{R(\cos x, \sin x, \tan x)}_{\text{some rational function involving } \cos x,\, \sin x,\, \tan x.}\,dx$$

Eg: $\int \frac{dx}{1+\sin x}$

Define: $\boxed{t = \tan\frac{x}{2}}$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx \Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} = \underbrace{\frac{2dt}{1+t^2}}_{\text{rational in } t}.$$

<u>Recall:</u> $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}.$

take $\theta = \frac{x}{2} \Rightarrow \tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2t}{1-t^2}.$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}.$$

$$\cos x = \frac{\sin x}{\tan x} = \frac{1-t^2}{1+t^2}.$$

In summary,

$t = \tan\frac{x}{2}$ then $dx = \boxed{\dfrac{2dt}{1+t^2}}$ and $\begin{cases} \tan x = \dfrac{2t}{1-t^2} \\ \sin x = \dfrac{2t}{1+t^2} \\ \cos x = \dfrac{1-t^2}{1+t^2} \end{cases}$

rational function of $t$.

<u>Eg.</u>
$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2t}{1+t^2}}\,\frac{2dt}{1+t^2}$$
$$= \int \frac{2dt}{(1+t^2)+2t}$$
$$= \int \frac{2dt}{(1+t)^2}$$
$$= \int \frac{2\,d(1+t)}{(1+t)^2} = -\frac{2}{1+t} + C$$
$$= -\frac{2}{1+\tan\frac{x}{2}} + C$$
※.

Eg
$$\int \frac{dx}{\sin^3 x} = \int \frac{1}{\left(\frac{2t}{1+t^2}\right)^3} \cdot \frac{2\,dt}{1+t^2}$$
$$= \int \frac{(1+t^2)^2}{4t^3}\,dt$$
$$= \frac{1}{4}\int \frac{1+2t^2+t^4}{t^3}\,dt$$
$$= \frac{1}{4}\int\left(\frac{1}{t^3}+\frac{2}{t}+t\right)dt$$
$$= \frac{1}{4}\left(-\frac{1}{2t^2}+2\ln|t|+\frac{1}{2}t^2\right)+C$$
$$= \frac{1}{4}\left(-\frac{1}{2\tan^2\frac{x}{2}}+2\ln\left|\tan\frac{x}{2}\right|+\frac{1}{2}\tan^2\frac{x}{2}\right)+C \quad \#.$$

## Improper Integrals

① closed & bdd ②

Best case scenario: $f: I = [a,b] \longrightarrow \mathbb{R}$ continuous

$\Rightarrow \int_a^b f(x)\,dx$ well-defined.

Fact: Sometimes ① or ② may fail to hold, but we may still be able to calculate $\int_a^b f(x)\,dx$.

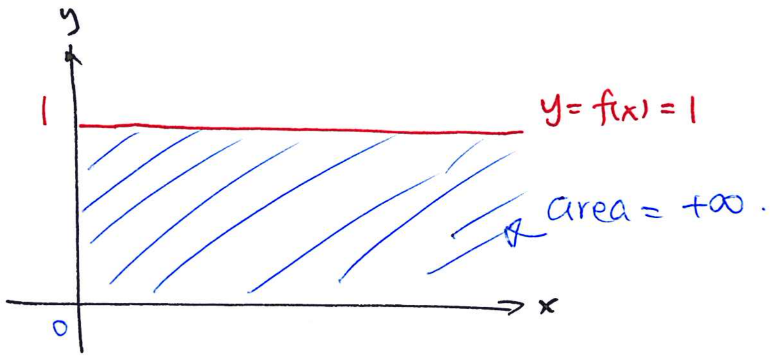
Type I Improper Integrals (e.g. $I$ infinite).

$$\int_a^{+\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{b} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\,dx.$$

Note: Some of these may not exist.

Eg. $f(x) \equiv 1$ then $\int_0^{+\infty} f(x)\,dx$ does not exist.

why? $\displaystyle\int_0^{+\infty} 1\,dx = x\Big|_0^{+\infty} = +\infty - 0 = +\infty$




We define the improper integrals using limits:

$$\int_a^{+\infty} f(x)\,dx := \lim_{t\to+\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

limit exists $\Rightarrow$ improper integral exists.

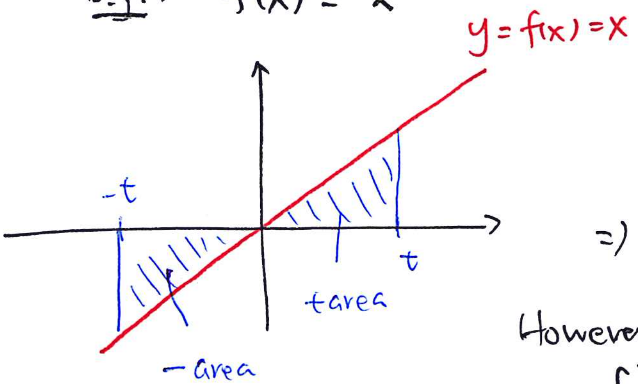
$$\int_{-\infty}^{+\infty} f(x)\,dx := \underbrace{\int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx}_{\text{exists and equal for ALL } c\in\mathbb{R}.}$$

Note: The last definition is a bit subtle.

cannot say that improper integrals exists

if we ONLY have $\underbrace{\displaystyle\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx \text{ exists.}}_{\text{why?}}$

E.g.: $f(x) = x$




$\displaystyle\int_{-t}^{t} f(x)\,dx = 0 \quad \forall t\in\mathbb{R}.$

$\Rightarrow \displaystyle\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx = 0.$

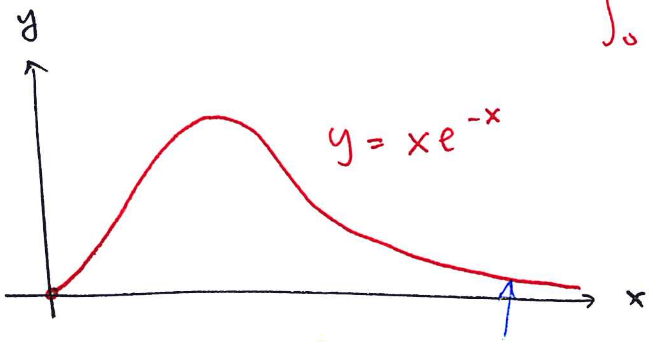
However,

$\displaystyle\int_0^{+\infty} f(x)\,dx$ NOT exist.

Examples:

① $\displaystyle\int_0^\infty x e^{-x}\,dx := \lim_{t\to+\infty}\int_0^t x e^{-x}\,dx.$ (if limit exists)

Observe: $\displaystyle\lim_{x\to+\infty} x e^{-x} = 0$. (not sufficient to say $\displaystyle\int_0^\infty x e^{-x}\,dx$ exists)




[ This needs to go to 0 fast enough.
otherwise $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots = +\infty$ ]

To be more precise, integrate by part.

$$\int_0^t \underbrace{x e^{-x}\,dx}_{-x\,d(e^{-x})} \overset{\dagger}{=} \left[-x e^{-x}\right]_0^t - \int_0^t e^{-x}\,d(-x)$$

$$= (-te^{-t} - 0) + \int_0^t e^{-x}\,dx$$

$$= -te^{-t} - \left[e^{-x}\right]_0^t$$

$$= -te^{-t} - (e^{-t} - 1).$$

$$\lim_{t\to+\infty}\int_0^t x e^{-x}\,dx = \lim_{t\to+\infty}\Big(\underbrace{-te^{-t}}_{\to 0} - \underbrace{e^{-t}}_{\to 0} + 1\Big) = 1.$$

So, $\displaystyle\int_0^\infty x e^{-x}\,dx = 1$ ※.

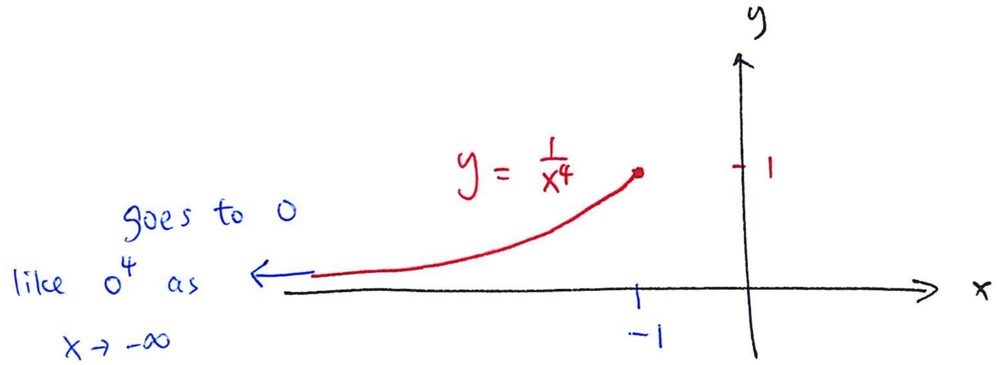
Question: $\displaystyle\int_{-\infty}^0 x e^{-x}\,dx$ does not exist

since $\displaystyle\lim_{x\to-\infty} x e^{-x} = -\infty \neq 0.$

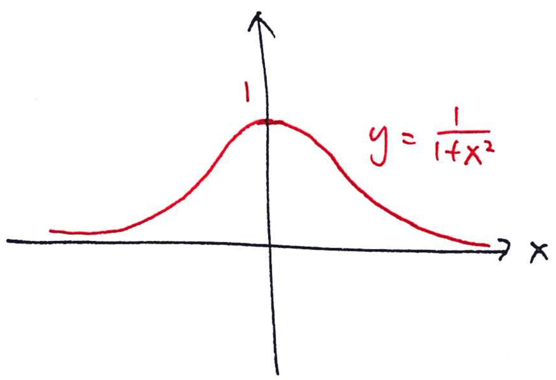
② $$\int_{-\infty}^{-1} \frac{1}{x^4}\,dx = \lim_{t\to-\infty} \int_{t}^{-1} \frac{1}{x^4}\,dx$$

$$= \lim_{t\to-\infty} \left[\frac{x^{-3}}{-3}\right]_t^{-1}$$

$$= \lim_{t\to-\infty} \left(\frac{1}{3} + \frac{1}{3t^3}\right) = \frac{1}{3} \;\#$$




③ $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \lim_{b\to+\infty} \int_a^b \frac{1}{1+x^2}\,dx$$




$$= \lim_{a\to-\infty} \lim_{b\to+\infty} \left[\tan^{-1}x\right]_a^b$$

$$= \lim_{b\to+\infty} \tan^{-1}b - \lim_{a\to-\infty} \tan^{-1}a$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \;.\#$$

<u>Theorem</u>: $\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p > 1$; is divergent if $p \le 1$

<u>Pf</u>: ~~$p>1$~~, $p \ne 1$ $$\int_1^t \frac{1}{x^p}\,dx = \left.\frac{-1}{(p-1)\,x^{p-1}}\right]_1^t = -\frac{1}{p-1}\left(\frac{1}{t^{p-1}} - 1\right).$$

$p>1$: $\lim_{t\to+\infty} = 0$ ✓ $\qquad$ $p<1$: $\lim_{t\to+\infty}$ <u>NOT</u> exist

<u>p=1</u>: $$\int_1^t \frac{1}{x}\,dx = \ln x\Big|_1^t = \ln t \to \infty \text{ as } t\to+\infty$$

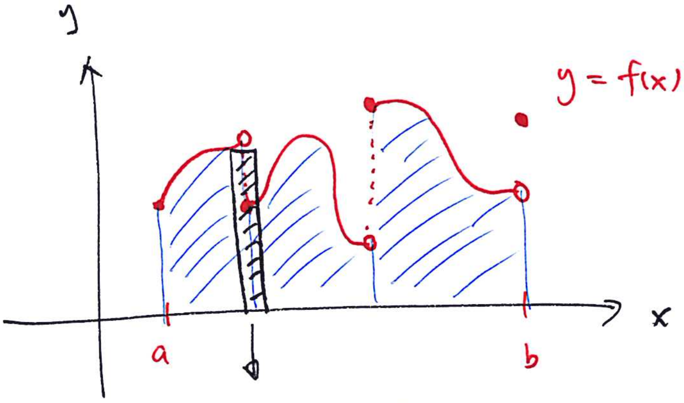
**Example:** $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$

**Sol:**
$$\int_2^t \frac{1}{x\ln x}\,dx = \int_2^t \frac{1}{\ln x}\,d(\ln x)$$
$$= \ln(\ln x)\Big|_{x=2}^{x=t}$$
$$= \ln(\ln t) - \ln(\ln 2).$$
$$\longrightarrow +\infty \quad \text{as } t \to +\infty$$

$\Rightarrow$ Improper integral does NOT exist.

**Type II Improper integrals:** $f$ is NOT cts at some point.

**Fact:** If $f: [a,b] \longrightarrow \mathbb{R}$ is bounded and is not continuous only at finitely many points, then the definite integral $\int_a^b f(x)\,dx$ still exists.




bad rectangles have area $\to 0$ as fineness $\to 0$

**Define:** (1) $f: (a,b] \longrightarrow \mathbb{R}$ is continuous.

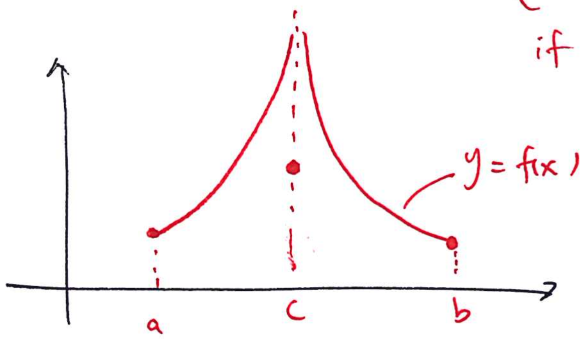
then $$\int_a^b f(x)\,dx := \lim_{t\to a} \underbrace{\int_t^b f(x)\,dx}_{\text{exist for every } t>a}. \quad \left(\begin{matrix}\text{if limit}\\ \text{exists}\end{matrix}\right)$$

(2) Similarly, if $f: [a,b) \to \mathbb{R}$ is continuous.
$$\int_a^b f(x)\,dx := \lim_{t\to b} \int_a^t f(x)\,dx.$$

(3) If $f : [a,b] \to \mathbb{R}$ is continuous except at $c \in (a,b)$

then
$$\int_a^b f(x)\,dx := \underbrace{\int_a^c f(x)\,dx}_{(2)} + \underbrace{\int_c^b f(x)\,dx}_{(1)}$$

if both exists.




**Examples:**

(1)
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0} \left. \frac{x^{1/2}}{1/2} \right|_t^1$$

cts for $x > 0$

$$= \lim_{t\to 0} \left( 2 - 2\sqrt{t} \right) = 2 \quad \#.$$

(2).
$$\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$$

cts except at $x = 1$
which lies in $[0,2]$

$$= \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^{2/3}} + \lim_{t\to 1^+} \int_t^2 \frac{dx}{(x-1)^{2/3}}.$$

$$= \lim_{t\to 1^-} \left. \frac{(x-1)^{1/3}}{1/3} \right|_0^t + \lim_{t\to 1^+} \left. \frac{(x-1)^{1/3}}{1/3} \right|_t^2$$

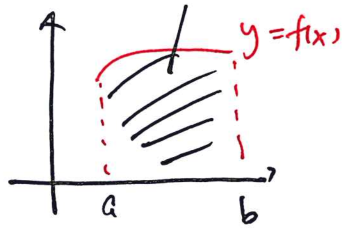
$$= \lim_{t\to 1^-} \left[ 3(t-1)^{1/3} - 3(-1) \right]$$
$$+ \lim_{t\to 1^+} \left[ 3(1) - 3(t-1)^{1/3} \right].$$

$$= 3 + 3 = 6 \quad \#.$$

Note:
$$\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \left. \frac{(x-1)^{1/3}}{1/3} \right|_0^2 = 6$$
(wrong logic).

• <u>Last time</u> ... t-substitution, improper integrals    Area $= \int_a^b f(x)\,dx$

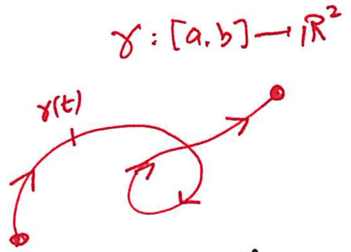
Def$^{\underline{n}}$ of definite integrals:




$$\int_a^b f(x)\,dx = \text{"signed" area under the curve } y=f(x) \text{ on } x\in[a,b]$$

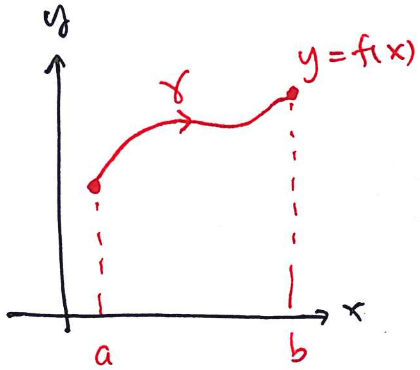
## <u>Applications to compute length / area / volume</u>

(1) Length of a curve $\gamma$ in a plane:



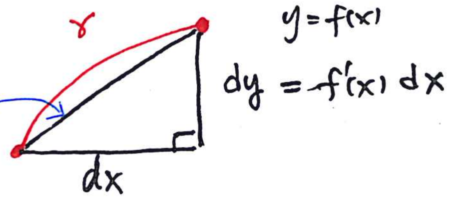

<u>Q</u>: What is the length of $\gamma$?

<u>A special case</u>: $\gamma$ = graph of $f(x)$.




$$\boxed{L(\gamma) = \int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

Look at a small portion of $\gamma$:



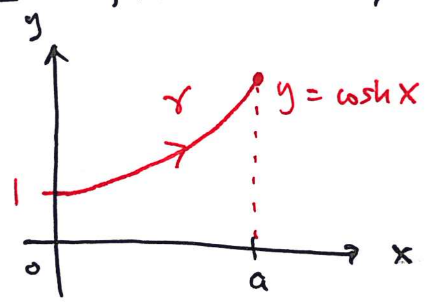

Pythagoras Thm:

length $= \sqrt{(dx)^2+(dy)^2}$

$= \sqrt{dx^2+(f'(x))^2dx^2}$

$ds = \sqrt{1+(f'(x))^2}\,dx \quad \Rightarrow L(\gamma) = \int_a^b ds = \int_a^b \sqrt{1+f'(x)^2}\,dx.$

<u>E.g.</u>: $f(x) = \cosh x, \quad x\in[0,a]$



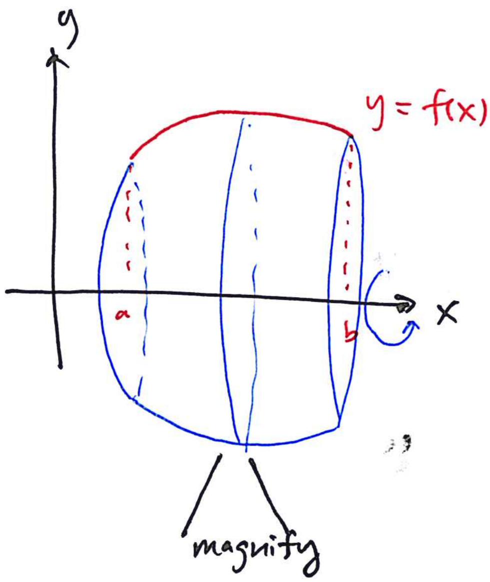

$$L(\gamma) = \int_0^a \sqrt{1+\sinh^2 x}\,dx$$

$$= \int_0^a \cosh x\,dx = \sinh x\Big|_0^a$$

$$= \sinh a$$ #

## (2) Area of a surface of revolution:

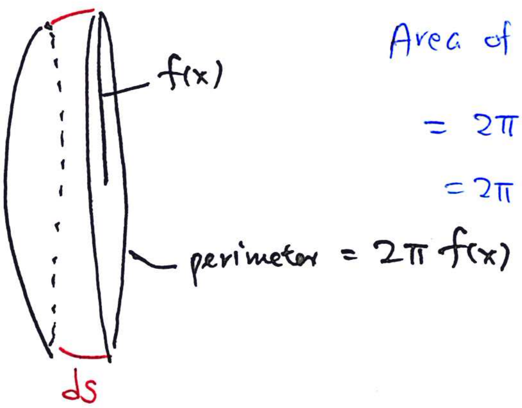
Surface of revolution:




Q: what is the area of the surface of revolution?

$$\boxed{\text{Area} = 2\pi \int_a^b f\sqrt{1+(f')^2}\,dx}$$

$$\text{“} = \int_a^b 2\pi \cdot f \cdot ds \text{”}$$

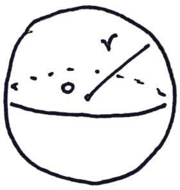


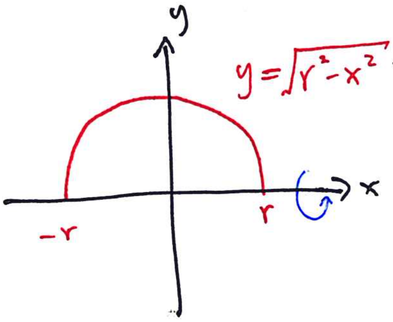

Area of this "cylinder"

$$= 2\pi\, f(x) \cdot ds$$

$$= 2\pi\, f(x)\sqrt{1+f'(x)^2}\,dx.$$

E.g.: Show that the area of a sphere of radius $r = 4\pi r^2$.




$y = \sqrt{r^2-x^2} = f(x).$

$$f'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}.$$

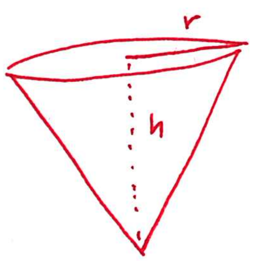
$$\text{Area} = \int_{-r}^{r} 2\pi f\sqrt{1+(f')^2}\,dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

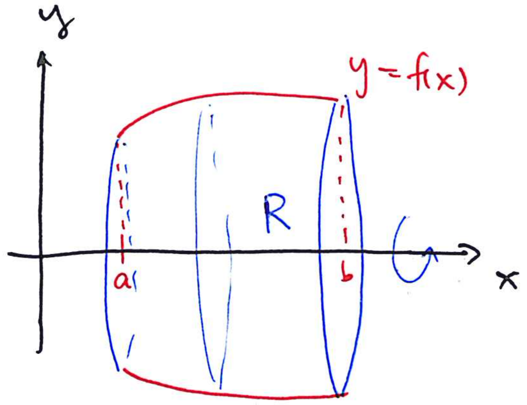
$$= 2\pi \int_{-r}^{r} \sqrt{(r^2-x^2)+x^2}\,dx$$

$$= 2\pi \int_{-r}^{r} r\,dx = 4\pi r^2 \;\#.$$

Ex: Calculate the area of a cone

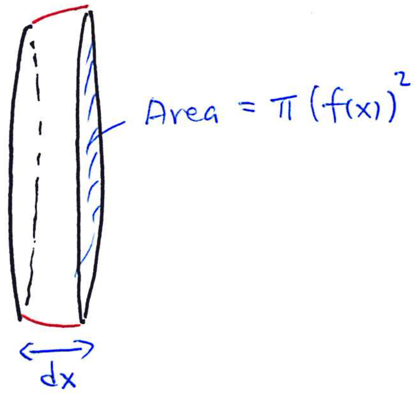


## (3) Volume of solid of revolution:




Q: What is the volume of the solid region R?

"slice up the solid"

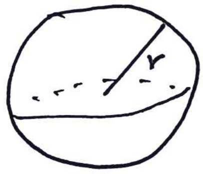



Volume $= \pi f(x)^2 dx$.

$\Downarrow$ integrate in $x$

$$\boxed{Vol(R) = \pi \int_a^b f(x)^2 dx.}$$

E.g.: Volume of a ball of radius $r$ $= \frac{4}{3}\pi r^3$.




Sol: $f(x) = \sqrt{r^2 - x^2}$, $x \in [-r, r]$.

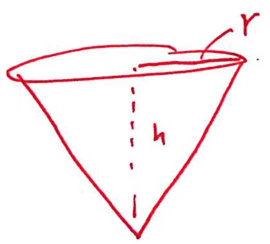
$$Vol = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \pi \left[ r^2 x - \frac{1}{3} x^3 \right]_{-r}^{r}$$

$$= \pi \left[ \left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right) \right]$$

$$= \pi \left( \frac{2}{3} r^3 + \frac{2}{3} r^3 \right) = \frac{4}{3}\pi r^3$$

Ex: Volume of the cone below?



Q: Is this all about "calculus"?

① **Differentiation**

$f'(x) = \text{slope}$

② **Integration**

$$\int_a^b f(x)\,dx = \text{area}$$

③ **Fundamental Thm**

$$\begin{cases} \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x) \\ \displaystyle\int_a^b F'(x)\,dx = F(b) - F(a). \end{cases}$$

Q: What about in higher dimensions?

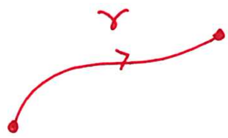
① If $f(x,y)$ has 2 parameters / variables $x, y$,

then how to "differentiate $f$"?

$\Rightarrow$ $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$. "partial derivatives", $Df$ : vector

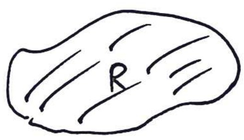
② $f(x,y)$ : how to integrate $f$ on

(a) a curve.




$$\int_\gamma f = ?$$

(b) a region




$$\iint_R f = ?$$

③ Fundamental Theorems :

- Divergence Thm
- Green's Thm
- Stokes' Thm

$\Rightarrow$ useful in Maxwell eq$^n$ in electromagnetism.